MATHEMATICS

MATHS PARTS I and II are comprehensive packages for pupils undertaking maths courses in the 13-17 year age range.

The package consists of :

1. Basic problems in maths with an almost infinite variation in the data provided.
2. Hints on revision.
3. A set of notes.

MATHS PART 1

SIDE A Program names

"ALG 1"
"ALG 2"
"ALG 3"
"REVISION"

SIDE B

"PLANE"
"SOLID"
"GEN"

"ALG 1" "ALG 2" "ALG 3"

Three areas are covered - functions, algebra and equations. You are expected to have revisied these areas thoroughly before attempting the problems. As a guide you should have covered all aspects of functions, mapping, reversed functions, solutions of equations, relationships, linear graphs, simultaneous equations, algebraic expressions, quadratic factors, factorisation, quadratic graphs, completing the square, algebraic manipulation and subjects of formulae.

Each of the three programs contains five different basic problem types giving a total of 15 problems all similar to those from past examination papers. Within each random numbers are generated to give a wide variety. Each program will go through each problem type twice to give ten questions. If your first answer is incorrect then you will be told the answer - there is no second chance in the exams - so check your working carefully. If you are wrong find out why. At the end you will be given a score out of ten. You should not feel happy about the thoroughness of your revision or your accuracy until you can consistently score ten on all three programs.

"REVISION"

This program contains advice gained from many years of teaching experience. There are comments for those who start early on a program of revision and for those panicking at the last minute.

"PLANE" "SOLID" "GEN"

Three programs are provided to give you experience in dealing with basic problems involving perimeters, areas, surface areas and volumes.

"PLANE" gives questions on the perimeters and areas of plane shapes. Similarly "SOLID" gives questions on the surface areas and volumes of all the common solids that you will meet. The third program "GEN" provides ten more general problems on the above topics. Within each program random numbers are again generated to give a wide variety. Each program will give you ten randomly selected problems. You need give the answer to one decimal place and it will be considered correct if within 0.1 of the actual answer. If your first answer is incorrect then you will be told the answer so please check your working carefully. If you are wrong again find out why. At the end you will be given a score out of ten. You should not feel happy until you can consistently obtain ten out of ten with this series of three programs.

REVISION NOTES

You will gain far more benefit by producing your own revision notes tailormade to your requirements, however much of this time is often spent extracting sections straight from books. A minimum of time being actually spent in learning and understanding the work.

To help you make your work more time effective we have produced a set of revision notes. In the text we have noted the pertinent points; we do expect you to add to this booklet and thus build up a set of notes to suit your needs.

LOADING INSTRUCTIONS

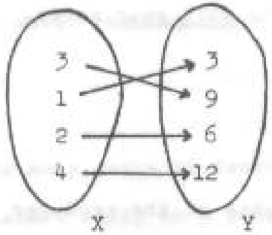
SPECTRUM 48K	LOAD ""
DRAGON 32	CLOAD
BBC	CHAIN ""

Details of other programs in this series and of other educational programs can be obtained from your software supplier or from:

SCISOFT 5 Minster Gardens, Newthorpe, Eastwood, NOTTS. NG16 2AT

FUNCTIONS

In this diagram of two sets X and Y a relationship is shown by the arrows



$1 \longrightarrow 3 \qquad 2 \longrightarrow 6$

$3 \longrightarrow 9 \qquad 4 \longrightarrow 12$

The number 1 in X <u>maps onto</u> the number 3 in Y

The number 3 in Y is the <u>image</u> of 1 in X

More generally:-

If $x \in X$ then $x \longrightarrow x \times 3$ or $x \longrightarrow 3x$

$x \longrightarrow 3x$ is a <u>rule</u> by which the image of X can be found

A rule can produce

(i) an image in Y for every element of X

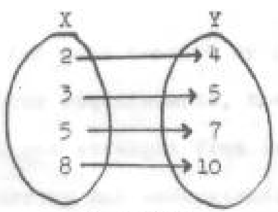
(ii) an image in Y for some values of X

(iii) more than one image in Y for some values of X

(iv) no image in Y for some values of X

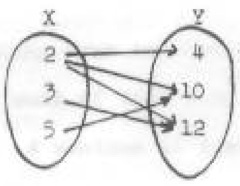
A rule which produces one and only one image in a set Y for each element of a set X is called a <u>function.</u>

Example:



$x \longrightarrow x+2$

(function)



$x \longrightarrow$ multiple of x

(not a function)

A function is normally referred to by a letter, the most common being f (and also g, h).

Thus

$f{:}x \longrightarrow 4x$ means f is the rule $x \longrightarrow 4x$

Using this rule, the image of 3 is 12 so $f{:}3 \longrightarrow 12$

Another way of writing this is $f(3) = 12$

or more generally $f(x) = 4x$

As another example if $g{:}x \longrightarrow \frac{1}{(x+4)}$ then $g(1) = \frac{1}{5}$

The normal rules of arithmetic are followed in calculating images.

Thus

$g{:}\ x \longrightarrow (x+2)\ 3$ is the same as

$g{:}\ x \longrightarrow (3x+6)$

Similarly

$g{:}\ x \longrightarrow (9x-81)$ is the same as

$g{:}\ x \longrightarrow 9(x-9)$

A <u>reversed</u> function can be written as

$2x+5 \longrightarrow x$

In this we have to find what value of x needs to be input to produce a specified output. Thus if we require an image of 19, we can write

$2x+5 = 19$

This is an <u>equation.</u>

The value of x that fits the equation is called the <u>solution.</u>

Finding the correct value for x is <u>solving</u> the equation.

Thus solving the equation $2x+5 = 19$ gives a solution of $x = 7$.

Solving the following equations gives the solutions stated:-

	Equations	Solutions
(i)	$\frac{1}{2}x-6 = 3$	$x = 18$
(ii)	$\frac{3x+2}{5} = 4$	$x = 6$
(iii)	$\frac{3(y+7)}{2} = 9$	$y = -1$
(iv)	$\frac{2}{5}c-6 = 0$	$c = 15$

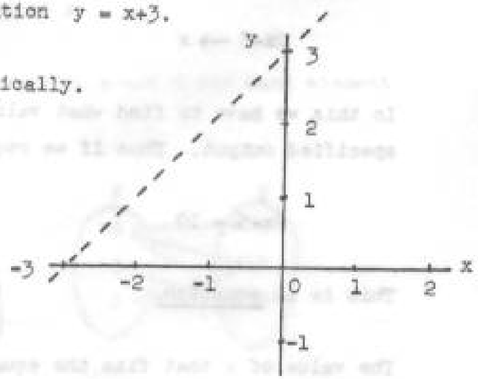
A function such as $f:x \longrightarrow x+3$ can also be represented as a set of ordered pairs.

Thus $L = \{(4,7), (2,5), (-1,2)...\}$ are some of the ordered pairs belonging to L.

If $(x,y) \in L$ then $y = x+3$

L is known as the solution set of the equation $y = x+3$.

The ordered pairs can be represented graphically.



The graph of $y = x+3$ cuts the x-axis at $(-3,0)$ and the y-axis at $(0,3)$

More generally a straight line is represented by the equation

$y = ax+b$

$(0,b)$ is the point at which the line cuts the y-axis (ordinate).

a is the slope or gradient of the line.

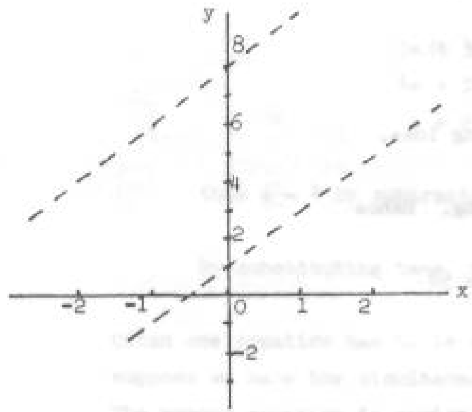
$(-\frac{b}{a},0)$ is the point at which the line cuts the x-axis (abscissae).

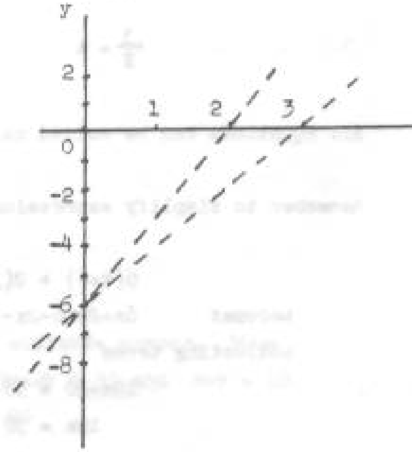
Thus two lines with the same b will intersect at $(0,b)$

and two lines with the same a will be parallel.

Plot $y = 2x+8$ $\quad$ $y = 3x-6$

$y = 2x+1$ $\quad$ $y = 2x-6$



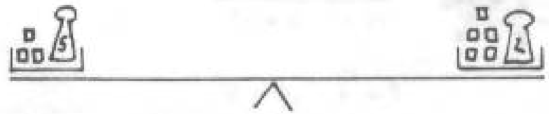


Many processes can be described in terms of an equation. For example, the length of time that it takes to discuss an item in a meeting is estimated as three times the number of people present, plus ten. This can be written as

$t = 3n+10$

This shows the relationship between n and t.

An equation can be used to describe the balance between two expressions. For example



shows us that if t is the weight of a tin then

$$3t+5 = 5t+2$$

The balance of the equation is preserved if the same arithmetical operation is carried out to both sides. Thus

$$5 = 2t+2 \qquad \text{(subtract 3t)}$$

$$3 = 2t \qquad \text{(subtract 2)}$$

$$\frac{3}{2} = t \qquad \text{(divide by 2)}$$

All equations can be solved using the balancing idea.

Remember to simplify expressions before solving. Hence

$$6(x+4) + 2(3-x) - 5(2-3x) = 58$$

becomes $6x+24+6-2x-10+15x = 58$

collecting terms

$$19x+20 = 58$$
$$19x = 38$$
$$x = 2$$

SIMULTANEOUS EQUATIONS

Two equations which are true statements at the same time are called a pair of simultaneous equations.

An ordered pair can make two equations true at the same time. Thus the ordered pair (6,2) makes the equations $T = 3D$ and $D = T-4$ true at the same time. It is the point where the graphs of the two equations intersect i.e. where $T = 6$ and $D = 2$.

Parallel lines have no point of intersection.

$$y = x+5$$
$$y = 2x$$

These two equations represent simultaneous equations. At the point of

intersection of the two lines, the values of x and y are the same for each equation. This occurs when

$$x+5 = 2x$$
$$\text{i.e.} \quad x = 5$$

When $x = 5$, $y = 10$ (checked in both simultaneous equations by substituting back)

Simultaneous equations can be solved in a different way. If we have

$$5a + 3b = 35$$
$$4a + 3b = 31$$
$$a + 0 = 4$$

then $a = 4$ by subtraction.

By substituting back $b = 5$.

Often one equation has to be multiplied by a suitable number. Thus suppose we have the simultaneous equations $9x+2y = 41$ and $x+y = 10$. The second equation is equivalent to $2x+2y = 20$ and we have

$$\begin{array}{r} 9x + 2y = 41 \\ \underline{2x + 2y = 20} \\ 7x + 0 = 21 \end{array}$$

$x = 3$ and $y = 7$

Further, it may be necessary to alter both equations in order to eliminate one of the letters

e.g. $5x + 3y = 19$ .....Eq(1)

$6x + 5y = 27$ .....Eq(2)

can be written

$25x + 15y = 95$	Eq(1) x 5
$\underline{18x + 15y = 81}$	Eq(2) x 3
$7x + 0 = 14$	Substract

$x = 2$, $y = 3$

FACTORS IN ALGEBRA

Algebraic expressions follow the normal arithmetical rules

Thus $(a+b)c = ac + bc$

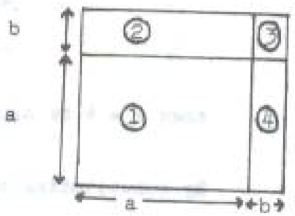
$a(x+y) + b(x-y) = ax + ay + bx - by$ (removing brackets)

$= ax + bx + ay - by$ (collecting like terms)

$= (a+b)x + (a-b)y$ (simplifying)

EXPRESSIONS TO KNOW:

$(a+b)^2 = a^2 + 2ab + b^2$



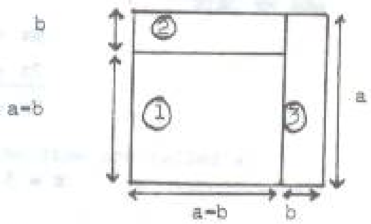
Proof: Total area = $(a+b) \times (a+b)$

Total area = area 1 + area 2
+ area 3 + area 4

$= a \times a + a \times b + b \times b + a \times b$

$= a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$



area 1 $= (a-b)^2$

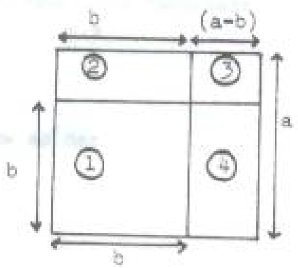
area 1 = Total area - area 2 - area 3

$= a \times a - b(a-b) - a \times b$

$= a^2 - ab + b^2 - ab$

$= a^2 - 2ab + b^2$

$a^2 - b^2 = (a+b)(a-b)$



$a^2 - b^2$ = total area - area 1

= area 2 + area 3 + area 4

$= b(a-b) + (a-b)(a-b) + b(a-b)$

$= (a-b)(b + a-b + b)$

$= (a-b)(a+b)$

This expression can be used to find the difference between two squares. For example

$$(1.6)^2 - (1.4)^2 = 3.0 \times 0.2$$
$$= 0.6$$

QUADRATIC FACTORS

An expression which contains a squared term is called a quadratic expression

e.g. $x^2 + 7x + 10$

This can be written as

$(x+5)(x+2)$ (check this)

and the expression is then said to be factorized

$(x+5)$ is one factor, $(x+2)$ is the other

The solution set for a quadratic equation are the values of x which satisfy the equation. Thus

$$x^2 + 7x + 10 = 0$$

has a solution set -5 and -2

Practise simplifying quadratic expressions

e.g. $(3x-5)(7x+4) = 21x^2 - 23x - 20$

and also factorizing quadratic expressions

e.g. $3x^2 + 7x + 4$

$(3x + 4)(x+1)$

QUADRATIC GRAPHS

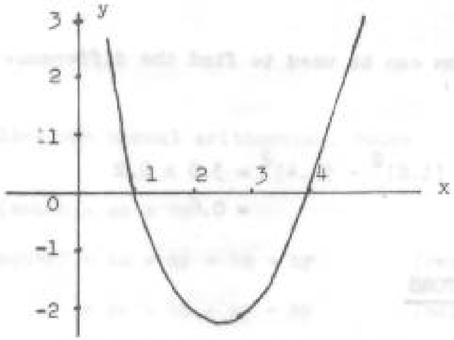
Quadratic graphs are represented by quadratic expressions

e.g. $f(x) = x^2 - 5x + 4$

Factorising gives

$$f(x) = (x-4)(x-1)$$

This will help in sketching the curve. By finding where $f(x)=0$, i.e. where the curve crosses the x-axis, it becomes easy to fix the general position of the U-shaped curve.



The curve for a quadratic expression with a positive x^2 term is U-shaped

The curve for a quadratic expression with a negative x^2 term is $\cap$-shaped.

Sometimes a quadratic expression can be written in the form

$$f(x) = (x-a)^2 + b$$

The associated curve can be obtained by translating $f(x) = x^2$ 'a' units to the right and 'b' units upwards.

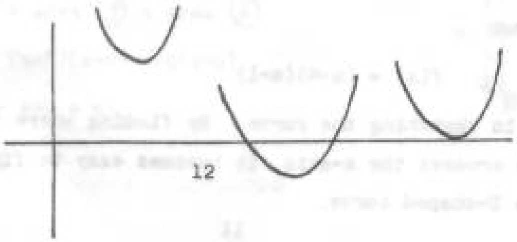
This method can also be used to find the solution to a quadratic equation

e.g.

$$\begin{aligned} x^2 - 6x + 5 &= 0 \\ x^2 - 6x &= -5 \\ (x-3)^2 - 9 &= -5 \\ (x-3)^2 &= 4 \\ x-3 &= +2 \text{ or } -2 \\ x &= 5 \text{ or } 1 \end{aligned}$$

This is known as <u>completing the square.</u>

A quadratic equation may have two, one or no elements in its solution set depending on whether the associated quadratic curve cuts, touches or does not cut the x-axis



For the general quadratic equation

$$ax^2 + bx + c = 0$$

the solution set is given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

<u>ALGEBRAIC MANIPULATION</u>

Manipulating symbols is an improtant aspect of mathematics.

In order to add algebraic fractions it is necessary to find the equivalent fractions which have a common denominator. These can then be added together.

e.g. 1. $\frac{a}{x} + \frac{b}{y} = \frac{ay}{xy} + \frac{bx}{xy} = \frac{ay + bx}{xy}$

e.g. 2. $\frac{2}{x-1} + \frac{3}{x-2} = \frac{2(x-2)}{(x-1)(x-2)} + \frac{3(x-1)}{(x-1)(x-2)}$

$$= \frac{2(x-2) + 3(x-1)}{(x-1)(x-2)}$$

$$= \frac{5x-7}{(x-1)(x-2)}$$

When an equation has one letter expressed in terms of other letters, the first letter is called the <u>subject of the formula</u>

e.g. $V = \pi r^2 h$ V is the subject

If the equation is rewritten then another letter can become the subject

e.g. $h = \frac{V}{\pi r^2}$ h is the subject

Manipulations can be complex and need care. They are sometimes necessary in solving equations.

For example find the solution set of

$$\frac{5}{(x-2)} + \frac{4}{(x-3)} = 2$$

Answer: $5(x-3) + 4(x-2) = 2(x-2)(x-3)$

$$9x - 23 = 2x^2 + 12 - 10x$$

$$2x^2 - 19x + 35 = 0$$

$$(2x-5)(x-7) = 0$$

$$x = \frac{5}{2} \text{ or } 7$$

Remember, always check your work carefully.

PERIMETERS AND AREAS OF PLANE SHAPES

The perimeter of a plane shape is the length of its boundary. For example, if the shape is a polygon (a closed plane shape with a finite number of straight edges), its perimeter is the sum of the lengths of its sides. The area of a plane shape is measured in square units.

When calculating areas, it is very important to make sure that all the linear dimensions are measured in the same units.

If the linear units are in cm, the area is given in sq.cm (or cm^2). Similarly if in mm, the area is given in sq.mm (or mm^2).

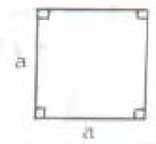
You will be required to know the following basic plane shapes but problems may combine two or more of these shapes.

(a) SQUARE

For a square of side a:

perimeter = 4a

area = a^2



e.g. For a square of side 3 cm, the perimeter = 12 cm and the area = 9 cm^2

(b) RECTANGLE

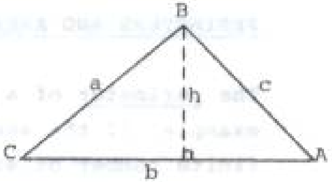
For a rectangle of length a and breadth b:

perimeter = 2b+2a = 2(b+a)

area = ba

e.g. For a rectangle of length 4 m and breadth 2 m, the perimeter = 12 m and the area = 8 m^2

(c) TRIANGLE



For a triangle with sides a,b,c and perpendicular height (as shown) h:

perimeter = a+b+c

area = ½bxh (½ base x perpendicular height)

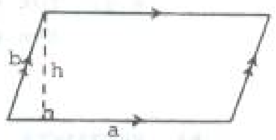
e.g. For a triangle with base 4 cm and perpendicular height 2 cm, the area = 4 sq.cm.
If the perpendicular height is not known, but two sides (a,b) and the included angle ($\angle C$) are, then the area of the triangle = ½ab sin C

If the lengths of all three sides are known then the area of the triangle = $\sqrt{S(S-a)(S-b)(S-c)}$ where S = ½(a+b+c).

e.g. For a triangle with sides measuring 6 cm, 5 cm and 3 cm the perimeter = 14 cm and the area
$= \sqrt{7(7-6)(7-5)(7-3)} = \sqrt{56} = 7.5$ sq.cm

(d) PARALLELOGRAM

A parallelogram is a quadrilateral with opposite sides parallel.



If the sides are of lengths a and b and the perpendicular height is h then:

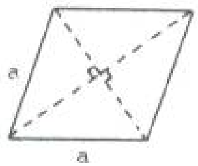
perimeter = 2(a+b)

area = ah (i.e. base x perpendicular height)

e.g. For example if the base is 10 cm and the perpendicular height is 5 cm then the area = 50 sq.cm

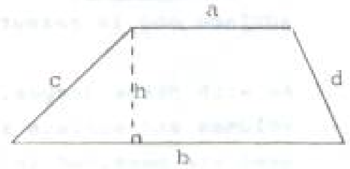
(e) RHOMBUS

A rhombus is just a special case of a parallelogram in which a = b i.e. all sides are equal.



(f) TRAPEZIUM

A trapezium is a quadrilateral with one pair of parallel sides.



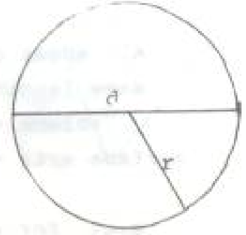
If the lengths of the sides are a,b,c and d and h is the perpendicular distance between the parallel sides then:

perimeter = a+b+c+d

area = ½(a+b)h

e.g. For example if the parallel sides of a trapezium have lengths 5 cm and 7 cm and the perpendicular distance between them is 2 cm then the area = 12 sq.cm

(g) CIRCLE



For a circle with radius r (diameter d = 2r) then:

perimeter = $2\pi r = \pi d$

area = πr^2

The value of π will normally be given as $3^1/7$ or 3.142.

e.g. For a circle of radius 7 cm, the perimeter is 44 cm and the area is 154 sq.cm.

VOLUME AND SURFACE AREA OF SOLID SHAPES

The volume of a solid shape is measured in cubic units.

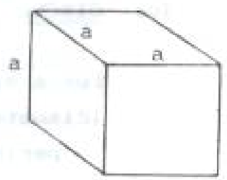
The surface area of a solid shape is the total area of its surface and is measured in square units.

As with plane shapes, it is very important when calculating volumes and surface areas to make sure that all the dimensions used are measured in the same units.

For example if the linear units are in cm, the surface area is in sq.cm (or cm^2) and the volume is in cu.cm (or cm^3).

You will be required to know the following basic solid shapes but problems may combine two or more of these shapes.

(a) CUBE



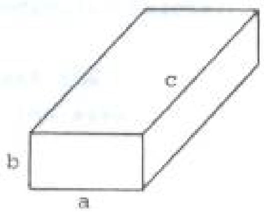
All edges of a cube are the same length. Let this be a.

volume = a^3

surface area = $6a^2$

e.g. For a square of side 3 cm, the volume = 27 cm^3 and the surface area = 54 cm^2

(b) CUBOID



A cuboid is a rectangular block. Let the lengths of the sides be a,b and c.

Then

volume = abc

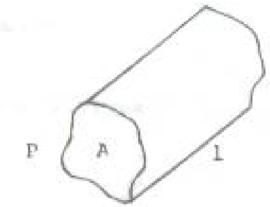
surface area = 2(ab+bc+ac)

e.g. For a cuboid with sides 2,3 and 4 cm,

volume = 24 cm^3

surface area = 52 cm^2

(c) PRISM



A prism is a solid with a uniform cross section.

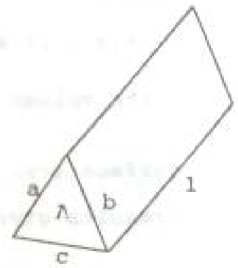
The prism shown has an end section of area A and perimeter P. The length is l.

volume = Al

surface area = 2A + pl

e.g. For a prism whose end face has perimeter 4 cm and area 2 cm^2 and whose length is 3 cm, then the surface area = 16 cm^2

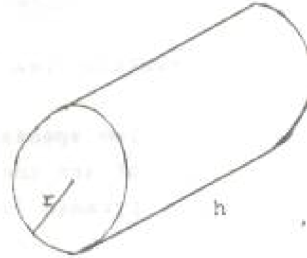
(d) TRIANGULAR-FACED PRISM



This is just a special case of a prism.

If the sides of the end face are of length 3, 4 and 5 cm and its area is 6 cm^2 and the length of the prism is 8 cm then the volume = 6x8 = 48 cm^3 and the surface area = 2x6 + 12x8 = 108 cm^2

(e) RIGHT CIRCULAR CYLINDER



This is again a special case of a prism.

If r is the radius of the end section (or base) and h is the height of the cylinder then

volume = $\pi r^2 h$

surface area = $2\pi r^2 + 2\pi rh = 2\pi r(r+h)$

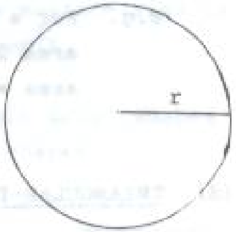
e.g. For a circular cylinder of height 10 cm and radius 7 cm

$$\text{volume} = \frac{22}{7} \times 7^2 \times 10 = 1540 \text{ cm}^3$$

$$\text{surface area} = 2 \times \frac{22}{7} \times 7^2 + 2 \times \frac{22}{7} \times 7 \times 10 = 308 + 440$$
$$= 748 \text{ cm}^2$$

(f) SPHERE

For a sphere of radius, r



$$\text{volume} = \frac{4}{3}\pi r^3$$

$$\text{surface area} = 4\pi r^2$$

e.g. If a sphere has radius 6 cm,

$$\text{the volume} = \frac{4}{3}\pi 6^3 = 904.9 \text{ cm}^3$$

$$\text{surface area} = 4\pi 6^2 = 452.4 \text{ cm}^2$$

(results given to 1dp using $\pi = 3.142$)

(g) PYRAMID

For a pyramid, let the area of the base be A and the perpendicular height be h then

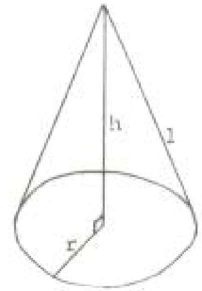
$$\text{volume} = \frac{1}{3}Ah$$

surface area = A + (area of triangular faces)

Two special cases of the pyramid which you should be aware of are the square-based pyramid and the triangular-based pyramid (usually termed a tetrahedron).

(h) RIGHT-CIRCULAR CONE

A right-circular cone is a cone with a circular base whose apex is directly above the centre of the circle.



If r is the radius of the circular base
h is the perpendicular height
l is the slant height,

$$\text{Then the volume of the cone} = \frac{1}{3}\pi r^2 h$$

$$\text{and the surface area of the cone} = \pi r^2 + \pi r l$$

e.g. If a right circular cone has radius 3 cm, height 4 cm and slant height 5 cm then

$$\text{the volume} = \frac{1}{3} \times \frac{22}{7} \times 9 \times 4 = 37.7 \text{ cm}^3$$

$$\text{and the surface area} = \frac{22}{7} \times 9 + \frac{22}{7} \times 3 \times 5 = 75.4 \text{ cm}^2$$